# Crossing of two mobile averages: A method for measuring the roughness exponent

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Self-affine signals characterized by a defined Hurst (roughness) exponent $H$ can be investigated through mobile averages. The density $\rho$ of crossing points between any two moving averages is a measure of long-range power-law correlations in the signal. The method is compared to the detrended fluctuation analysis. We take advantage of our findings in order to propose a practically powerful and accurate technique for determining $H$ and apply it right away to cases with persistent or antipersistent correlations. [S1063-651X(98)10411-7]

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Stochastic processes and mechanisms that generate fluctuating sequences are numerous [1]. Conversely, it is relevant to characterize natural signals. Among numerous proposed techniques, one consists in measuring the roughness (self-affine) exponent of a signal. Roughness or Hurst exponents are commonly measured in surface science [2] and time series analysis [3].

We propose a technique based on the so-called mobile averages that will be found to be very powerful and accurate, and contain original physical meanings, even though much still has to be done to satisfy scientific rigor and establish the concept. In the concluding paragraphs we will open up directions toward new investigations.

Consider a time series $y(t)$ given at discrete times $t$. At time $t$, the mobile average $\overline{y}$ is defined as

$$\overline{y}=\frac{1}{T}\sum_{i=0}^{T-1} y(t-i), \tag{1}$$

i.e., the average of $y$ for the last $T$ data points. One can easily show that if $y$ increases (decreases) with time, $\overline{y}<y$ ($\overline{y}>y$). Thus, the mobile average captures the trend of the signal over a time interval $T$. Such a procedure can be used in fact on any time series, as in atmospheric or meteorological data, DNA, financial data fracture, internet, traffic, and fractional Brownian motions.

Consider two different mobile averages $\overline{y_1}$ and $\overline{y_2}$ characterized respectively over, e.g., $T_1$ and $T_2$ intervals such that $T_2>T_1$. These mobile averages are illustrated in Fig. 1 in the specific case of the evolution of a typical stochastic motion and for $T_1=5$ and $T_2=30$. The crossings of $\overline{y_1}$ and $\overline{y_2}$ coincide with drastic changes of the trend of $y(t)$. If $y(t)$ increases for a long period before decreasing rapidly, $\overline{y_1}$ will cross $\overline{y_2}$ from above. This event is called a ''death cross'' in empirical finance [4]. On the contrary, if $\overline{y_1}$ crosses $\overline{y_2}$ from below, the crossing point coincides with an upsurge of the signal $y(t)$. This event is called a ''gold cross.'' Financial analysts often try to ''extrapolate'' the evolution of $y_1$ and $y_2$ expecting ''gold'' or ''death'' crosses. Most computers on trading places are equipped for performing this kind of analysis and forecasting [5]. Obviously, the forecasting in empirical finance is based on ''recipes'' that are specific to the considered market. Even though mobile averages seem to be ''artificial'' measures, we will see below that they present some very practical interest for physicists and raise new questions.

One aim in the present report is to characterize the basic statistics of the crossing points of mobile averages on time series exhibiting long-range power-law correlations. Physicists will understand the opportunities behind the findings and will be able to imagine the usual routes of useful investigations after the first basic idea presented.

The artificial time series used for the following demonstration within the successive random addition method originates in $d=1$ landscape profile construction. This method is also called ''midpoint displacement'' in the literature [6]. With this algorithm based on iterations, one generates a sequence of length $N=2^n+1$ where $n$ is an iteration number. At each iteration, one finds the intermediate positions (midpoints) of couples of neighboring points and calculates the values of the signal at the midpoints through some interpolation with respect to neighboring couples. The midpoint values are then displaced by random numbers chosen from a normal distribution with zero mean and variance $\sigma^2/2^{2nH}$. The parameter $H$ is the Hurst exponent of the resulting self-affine signal or fractional Brownian motion. For such a (discrete) self-affine signal $y(t)$, we can choose a particular

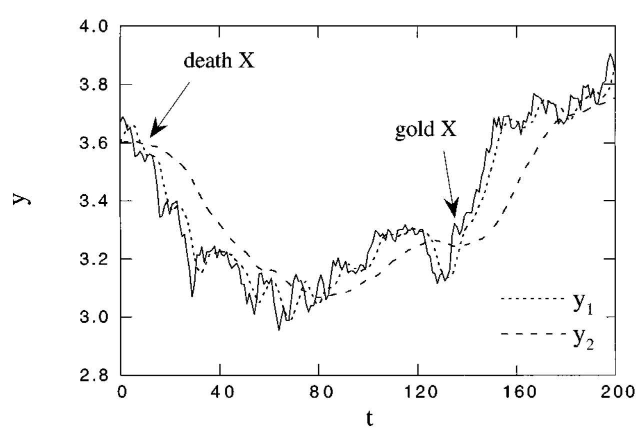


FIG. 1. Two moving averages $\overline{y_1}$ and $\overline{y_2}$ of an arbitrary signal for $T_1=5$ and for $T_2=30$. A ''gold cross'' and a ''death cross'' are denoted (see definition in the main text).

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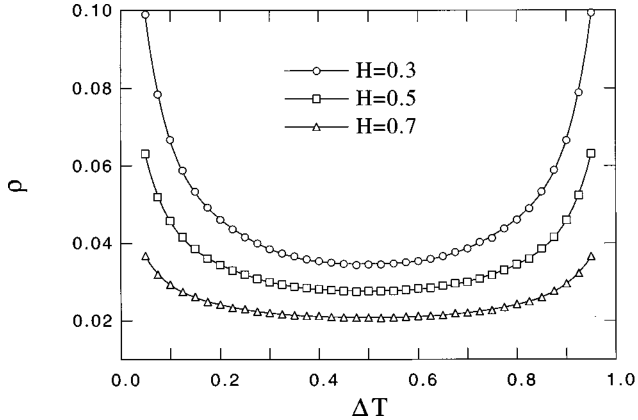


FIG. 2. The density $\rho$ of crossing points as a function of the relative difference $\Delta T$ with $T_2=80$. Different values of $H$ are illustrated: $H=0.3$, 0.5, and 0.7.

point on the signal and rescale its neighborhood by a factor $b$ using the roughness (or Hurst [8,3]) exponent $H$ and defining the new signal $b^{-H}y(bt)$. For the correct exponent value $H$, the signal obtained should be indistinguishable from the original one, i.e.,

$$y(t)\sim b^{-H}y(bt). \tag{2}$$

An exponent $H<1/2$ implies an *antipersistent* behavior while $H>1/2$ means a so-called *persistent* signal [3]. The simple Brownian motion is characterized by $H=1/2$ and white noise by $H=0$. Using the *successive random addition* technique described above, we have built time series up to $N=262\,145$ data points ($n=18$ iterations).

It is well known [3] that the set of crossing points between the signal $y(t)$ and the $y=0$ level is a Cantor set with a fractal dimension $1-H$. The related physics pertains to so-called studies in first return time problems [9]. However, the question can be raised whether there is a Cantor set for crossing points between $\overline{y_1}$ and $\overline{y_2}$. We have calculated the density of such crossing points $\rho$ for various values of $H$. In all checked cases, $\rho$ is independent of the size $N$ of the time series. In so doing, the fractal dimension of the set of crossing points is 1, i.e., the points are homogeneously distributed in time along $\overline{y_1}$ and $\overline{y_2}$. Due to the homogeneous distribution of crossing points, the forecasting of ‘‘gold’’ and ‘‘death’’ crosses is impossible even for self-affine signals $y(t)$. This is different from the forecasting of the sign of each fluctuation $y(t+1)-y(t)$ which is possible for $H\neq 1/2$.

When $T$ is large, $\overline{y}(t)$ is smooth and ‘‘relatively distant’’ from the signal $y(t)$ while for small $T$ values, $\overline{y}(t)$ rather follows the excursion of the signal. Thus, it is of high interest to observe how $\rho$ behaves and would have some scaling behavior with respect to the relative difference $T_1-T_2$. More precisely, consider the relative difference $0<\Delta T<1$ defined as

$$\Delta T=\frac{T_2-T_1}{T_2}. \tag{3}$$

Figure 2 presents on linear scales the plot of $\rho$ as a function of $\Delta T$ for $H=0.3$, 0.5, and 0.7. The parameter $T_2$ was fixed to be 80. The $\rho(\Delta T)$ curve is fully symmetric and diverges for $\Delta T=0$ and for $\Delta T=1$, i.e., for identical $\overline{y_1}$ and $\overline{y_2}$. For

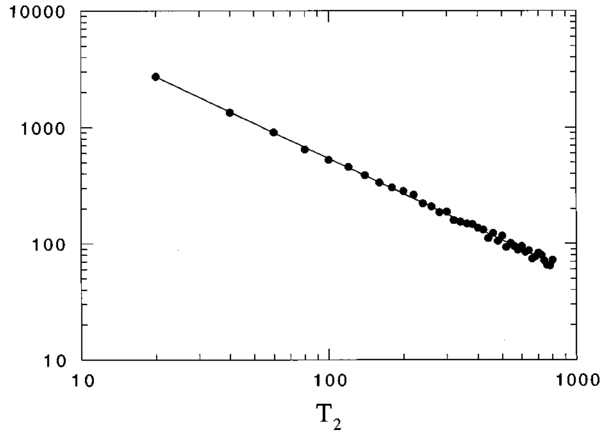


FIG. 3. The density of crossing points $\rho$ between two mobile averages of a $H=0.7$ signal as a function of $T_2$ illustrated for $T_1=T_2/2$.

$T_1=T_2/2$, the density of crossing points has a minimum. It should be noted that this remarkable result was not mentioned elsewhere probably due to the fact that some theoretical framework for the mobile average method is missing.

Moreover, for small $\Delta T$ values, we find that $\rho$ scales as $\Delta T^{H-1}$ as well as $\rho\sim(1-\Delta T)^{H-1}$ for $\Delta T$ values close to 1. We have also found that $\rho$ scales as $T_2^{-1}$. This is illustrated in Fig. 3 by the plot of $\rho(T_2)$ for a $H=0.7$ signal and for $T_1=(T_2/2)$, i.e., the minimum of the $\rho(\Delta T)$ curve as a function of $T_2$. Considering the above behaviors, we propose the general form for the density of crossing points

$$\rho\sim\frac{1}{T_2}\,[(\Delta T)(1-\Delta T)]^{H-1}. \tag{4}$$

Two time scales appear in Eq. (4): $\Delta T$ and $T_2$. The time difference $\Delta T$ allows for the investigations of the correlations ($H$) lying in the signal. The largest period $T_2$ controls trivially the amplitude of $\rho$: the greater is $T_2$, the smoother is the mobile average $y_2$, and the fewer is the number of crossing points. The above relationship is quite similar to the density of states on a fractal lattice, which are solutions of the Schrödinger equation [10]. It is analogous to the age distribution of domains in coarsening problems in the spinlike model [11].

One practical interest of the above findings stems in the easy implementation of an algorithm for measuring $H$. We have tested the accuracy of the estimation of $H$ through the measure of $\rho$ on time series of length $N=2^{13}+1$ and we have varied the parameter $H$. Figure 4 presents the relative error (%) in determining $H$ for the present technique as a function of $H$. We consider here only the case $T_2=100$. Two main advantages of the technique have to be mentioned: (i) the algorithm is fast and can be implemented in real-time analysis; (ii) the technique is insensitive to local and global misleading trends that the signal may exhibit. This ‘‘insensivity’’ is due to the fact that similar trends appear in both $\overline{y_1}$ and $\overline{y_2}$.

There are many ways to measure the Hurst exponent and the related fractal dimension $D_f=2-H$ of a scalar time series. The above measure of $H$ can be compared with that obtained on the same time series using the detrended fluctuation analysis (DFA), which is also insensitive to local and

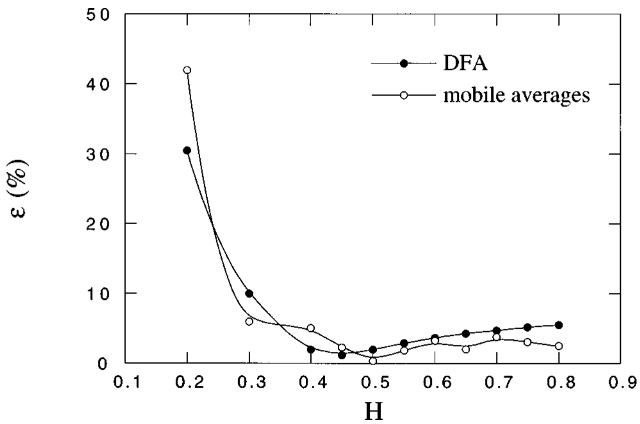


FIG. 4. Relative error $\epsilon$ (in %) for the estimation of the measured Hurst exponent by DFA and mobile averages as a function of the Hurst exponent $H$ of artificial time series constructed by the midpoint displacement method.

global trends. In short, DFA consists in dividing a random variable sequence $y(n)$ of length $N$ into $N/t$ nonoverlapping boxes, each containing $t$ points. The best linear trend $z(n) = an+b$ in each box is defined. The fluctuation function $F(t)$ is then calculated following

$$F^2(t)=\frac{1}{t}\sum_{n=(k-1)t+1}^{kt}|y(n)-z(n)|^2, \quad k=1,2,\ldots,N/t. \tag{5}$$

Averaging $F(t)$ over the $N/t$ intervals gives the fluctuations $\langle F(t)\rangle$ as a function of $t$. If the $y(n)$ data are random uncorrelated variables or short range correlated variables, the behavior is expected to be a power law

$$\langle F\rangle \sim t^H. \tag{6}$$

The results of DFA are also illustrated in Fig. 4 after [7].

As observed in Fig. 4, the accuracy of the mobile average method is of the same order than the one of the DFA. It should be noted that the relative error in the determination of $H$ is huge for small $H$ values due to the stationarity of signal. Improved methods are then needed [7].

One should also remark that the present technique can be easily extended to the case of multifractal (or multiaffine) signals, which are more elaborate than the present self-affine signals. It is indeed possible to investigate the crossing of the various moments $q$ of the signal $y$ calculated over two different periods $T_1$ and $T_2$ ($q$ is fixed to 1 herein). The whole multifractal $H(q)$ spectrum should then be found. This is outside the scope of the present report and is planned to be examined in the near future.

In summary, we have investigated the density $\rho$ of crossing points between two mobile averages on a self-affine signal $y$. As a function of the Hurst exponent $H$ characterizing the self-affinity, it is found that $\rho$ is a symmetric function of $\Delta T$, i.e., the difference between the periods of the averages. Moreover, the behavior of $\rho$ depends only on the roughness $H$ of the signal $y$. It turns out that the crossing points of two mobile averages can be used in order to determine the Hurst exponent of a self-affine signal. It seems that this technique gives accurate values of $H$ as good as those obtained with the DFA technique, and suggests interesting developments.

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